Application of L^1 Reconstruction of Sparse Signals to Ambiguity Resolution in Radar

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***Abstract*—A novel approach for range-Doppler ambiguity resolution in pulse Doppler radars is presented. The new technique makes use of the sparse measurement structure of the post-detection data in multiple pulse repetition frequency radars and the resulting L^0/L^1 equivalence. The ambiguity resolution problem is cast as a linear system of equations which is solved for the unique sparse solution. Numerical results show that the ambiguities can be effectively resolved even with the number of measurements much less than the number of unknowns. The proposed technique reduces the number of PRFs required to resolve multiple targets in some cases compared to conventional techniques.**

I. INTRODUCTION

The broad class of pulse Doppler radars can be used to simultaneously estimate the range and radial velocity of multiple targets in the radar's field of view. However, these measurements are subject to aliasing ambiguities in both range and Doppler, making it difficult to determine the correct, unaliased range and Doppler shift of detected targets [1]. As an example, range ambiguities can exist when a second pulse is transmitted before the most distant detectable echo from a previous pulse has been received. When that echo is received and detected, the processor does not know if it represents a target from the most recent pulse and a relatively short range, or the earlier pulse and a longer range. If the time between pulses is T (commonly called the *pulse repetition interval* (PRI) or *inter-pulse period* (IPP)), the *unambiguous range* is $R_{ua} = c\cdot T/2$, where c is the speed of light (about 3×10^8 m/s) and the corresponding frequency $1/T$ is called the *pulse repetition frequency* (PRF). The apparent range R_a becomes the actual range measured modulo R_{ua}: $R_a = R$ modulo R_{ua}. It is then necessary to resolve the ambiguity (*disambiguate* the measurement), that is, determine R given R_a and R_{ua}. Unfortunately, the answer is not unique: there are multiple ranges of the form $R_n = R_{ua} + n\cdot R_{ua}$ that are consistent with the measurement R_a. The identical problem exists with Doppler (velocity) measurements, since the actual Doppler shift will be aliased into the interval $(-PRF/2,+PRF/2)$.

The range/Doppler ambiguity resolution problem is conventionally addressed by repeating the basic pulse Doppler measurement with several different pulse repetition frequencies, producing different aliasing characteristics. Each of the PRFs yields ambiguous measurements but the combined measurements from a well-chosen set of PRFs can eliminate all ambiguities out to distances given by the radar's sensitivity or beyond. One of several algorithms is then applied to determine the true range/velocity pairs that are consistent with all the measurements. Common algorithms include the Chinese Remainder Theorem (CRT) and the "coincidence" algorithm. The Chinese remainder theorem is an analytic procedure for calculating the unambiguous range from the measured range using multiple *PRFs*, and the coincidence approach is a graphical application of the basic CRT principle [1], [2].

In practice, detection maps are sparse: detections exist in only a few range/velocity bins. L^1 optimization methods have recently been shown to be very effective in developing sparse solutions to various sensing problems. The objective of this paper is to investigate whether L^1 techniques can provide an improved means of range/velocity ambiguity resolution.

II. MATHEMATICAL MODEL OF THE MEASUREMENT PROCESS

In this paper, we consider post-detection ambiguity resolution in both range and Doppler. "Post detection" means that the data is examined after threshold detection [1] has been performed. At this point, the data has two important characteristics. First, it is binary: in each range-Doppler bin, either a target was detected at that location, or it was not. The outcome of the detection process after threshold detection can, therefore, be represented as either a "1" or a "0" for each range-Doppler bin tested, "1" representing the presence of a target and "0" representing the case when it is decided that the received echoes in the bin under test are most likely due to interference only and no target is detected. Second, the measured data after detection is *sparse*. In real environments, most range or Doppler bins will not contain a detectable target; detections will be present in only a small fraction of the bins. Our goal is to develop new algorithms for range and Doppler ambiguity resolution in radar detection data using L^1 minimization methods for sparse signals, and to investigate the properties of such techniques. We assume a standard multi-PRF measurement protocol, and concentrate on a new solution method for the sparse binary equations.

By way of example, consider a system transmitting pulses with a PRI corresponding to $R_{ua} = 4$ range bins. We will first state the problem in terms of range, but all of the results apply equally well in the Doppler dimension. Assume the sensitivity

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of the radar is such that targets could possibly be detected at ranges out to the 7th range bin.[1] Now suppose that in fact there are detectable targets at ranges corresponding to the 2nd and 7th range bins. For the second and subsequent pulses, these will result in detections that appear to occur in range bins 3 (bin 7 modulo 4) and 2 (bin 2 modulo 4). This measurement process can be represented as the following $m \times n$ set of linear equations, where the number of unknowns n is the number of range bins over which targets might be detected, and the number of measurements m is the number of range bins in the unambiguous range interval:

$$\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1&0&0&0&1&0&0\\0&1&0&0&0&1&0\\0&0&1&0&0&0&1\\0&0&0&1&0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\0\\0\\1\end{bmatrix} \tag{1}$$

For a single PRF, $n > m$ so that the solution is under-determined.

Notice that the matrix describing this system is a 4×7 binary Toeplitz matrix. Denoting this matrix as $T_{m,n}$, Eq. (1) becomes simply

$$\mathbf{y} = T_{4,7}\mathbf{x}. \tag{2}$$

If the measurements were repeated with a different PRF, e.g. one with 5 bins in the unambiguous range interval, a new system of equations described by a 5×7 binary Toeplitz matrix would be generated.

III. PROBLEM FORMULATION FOR MULTI-PRF MEASUREMENTS

The measurements for a given PRF can be expressed in the form of a Toeplitz matrix applied to the actual target distribution in range or Doppler as shown in Eq. (1). The individual Toeplitz matrices for different PRFs can then be vertically concatenated to form the measurement matrix for a linear system of equations that represents all of the measured data. Depending on the number of PRFs, the number of unaliased range bins which is given by n, and the sum of the number of range bins in the unambiguous range for each of the PRFs representing the total number of measurements m_s, the resulting system of equations may be either over- or under-determined.

Continuing with the same example, consider a three-PRF case with the truth vector $\mathbf{x} \in \mathcal{B}^{13}$ where $\mathcal{B}$ denotes a binary space, and the three PRFs corresponding to unambiguous ranges of 4, 5 and 6 range bins. We therefore have the following three sets of equations:

$$\begin{aligned}\mathbf{y_4} &= T_{4,7}\mathbf{x}\\ \mathbf{y_5} &= T_{5,7}\mathbf{x}\\ \mathbf{y_6} &= T_{6,7}\mathbf{x}.\end{aligned} \tag{3}$$

We form a single system of linear equations $\mathbf{y} = A\mathbf{x}$ with measurement matrix A ($m_s \times n$) by concatenating the three Toeplitz matrices together:

$$\mathbf{y} = A\mathbf{x} = \begin{bmatrix}T_{4,7}\mathbf{x}\\T_{5,7}\mathbf{x}\\T_{6,7}\mathbf{x}\end{bmatrix} \tag{4}$$

A is then a binary $\mathcal{B}^{15\times 7}$ matrix with 15 linear measurements of the unknown vector **x**. The solution to the over-determined system of equations so formulated is the truth vector **x** with unambiguous range information. It is, however, the under-determined case with the number of measurements much less than the number of unknowns ($m_s \ll n$) that is of more practical interest and hence will be discussed next in detail.

The ambiguity resolution problem can be easily extended in another dimension to account for both range and Doppler ambiguities by vectorizing the range-Doppler grid. Once the dimensions of the range-Doppler grid have been defined – the range bins based on the range swath and range resolution, and the Doppler cells based on the PRFs and the estimated range of Doppler shifts and Doppler resolution – we form the range and Doppler binary Toeplitz matrices T_R and T_D respectively in the same way as shown in Eq. (1). T_R and T_D are combined for each PRF into one binary matrix $A_{\mathbf{PRF}}$ by taking the Kronecker product of T_R and T_D. The order of the Kronecker product is dictated by the order in which the range-Doppler grid is vectorized: if the fast time range samples are collected one column after another and put in a single vector to form the truth vector x, then

$$A_{PRF} = T_D \otimes T_R, \tag{5}$$

where $\otimes$ represents the Kronecker product. As in the one dimensional case, a single system of linear equations is then formed by vertically concatenating the individual A_{PRF} matrices for multiple PRFs to form the measurement matrix A ($m_s \times n$) of the combined system that represents all of the measured data as shown in Eq. (4). The total number of measurements in the concatenated data m_s is now given by adding up the products of the number of range cells in the unambiguous range and the number of velocity cells in the unambiguous velocity measurements of each PRF, and the total number of unknowns n is now the product of the number of range bins and the number of Doppler cells over which the target might be detected.

The mathematical model described in section II results in a non-binary observation vector **y** in the case when two targets alias to the same location in **y**, a phenomenon we refer to as a *collision*. Specifically, if two targets alias to the same bin in the observation vector, that measurement will have a value of two

[1] These ranges are unrealistically short, but are convenient for writing out explicit equations. Real radars might have PRIs corresponding to hundreds to a few thousand range bins, and be sensitive enough to detect targets at those ranges or further.

instead of one, so the measurement vector is no longer binary. This is not possible in the normal radar threshold detection process, which cannot distinguish when a threshold crossing is due to one or multiple coincident targets and therefore produces only a 0 or a 1 in the measurement vector. To model this behavior, if two distinct targets alias to the same apparent range bin, the integer values greater than 1 are clipped to a value of 1 representing the practical case of single target detection for a single threshold crossing. The clipping, however, results in a case of measurement error which must be dealt with for the correct unambiguous detection of all targets. Additional sources of measurement errors such as false alarms and missed detections and their effects on the ambiguity resolution process are the subject of future research.

IV. Application of L^1 Minimization Techniques in Ambiguity Resolution

With the prior knowledge that the measured data is sparse, the under-determined system representing the post-detection aliased measurements can be solved for the minimum L^0 "norm" solution, i.e. a solution with minimum number of non-zero components, as an attempt to obtain the unambiguous truth vector $\mathbf{x}$:

$$\mathbf{x} = \min \|\mathbf{x}\|_0 \text{ such that } \mathbf{y} = \mathbf{A}\mathbf{x}, \quad (6)$$

where $\|\mathbf{x}\|_n$ represents the L^n norm of $\mathbf{x}$. However, finding the minimum L^0 norm solution is a well-known Np-hard problem because of its combinatorial nature. The L^1 norm solution, on the other hand, is a convex optimization problem and can be solved using techniques available in the literature. Fortunately, it has been shown that the minimum L^0 norm solution for any given unknown vector $\mathbf{x} \in \mathcal{R}^N$ is also given by the minimum L^1 norm solution, provided $\mathbf{x}$ is sufficiently sparse. Several studies have been conducted to establish the sparsity bound for L^0/L^1 equivalence in recent years [4]-[6]. Donoho [5] defines the *Equivalence Breakdown Point* of a matrix Φ, $EBP(\Phi)$, as the "maximal number N such that, for every α_0 with fewer than N non-zeros, the corresponding vector $S = \Phi\alpha_0$ generates a linear system $S = \Phi\alpha$ for which the L^0 and L^1 norm problems have identical unique solutions, both equal to α_0". The EBP typically exceeds $O(n/\log_2(m_s))$. Furthermore, empirical examples were shown by Candès, Romberg and Tao [4] where equivalence held with as many as $n/4$ non-zeros for random partial Fourier measurement matrices.

As shown in Section III, the measurement matrix for our multi-PRF system is formed by concatenating the individual binary Toeplitz matrices representing a single PRF. It can be seen that the measurement matrix so formed has two important properties:

1) It is non-negative.
2) The sum of the elements in each column is the same, and equals the number of PRFs.

The measurement matrix in the two dimensional range-Doppler case also possesses these properties. L^1 recovery with measurement matrices which possess these two features was considered by Khajehnejad, Xu, Dimakis and Hassibi [17] and it was proved that if $A \in \mathcal{R}^{m\times n}$ is "a matrix with non-negative entries and constant column sum, then for all non-negative k-sparse $\mathbf{x}_0$, it holds that

$$\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}_0, \mathbf{x} \geq 0\} = \{\mathbf{x}_0\}, \quad (7)$$

i.e. the condition for the success of L^1 recovery reduces to simply the condition for there being a "unique" vector in the constraint set". Exploiting this key fact, under-determined systems representing radar measurements of the sparse target distribution in range or Doppler can be solved for the minimum L^1 norm solution to determine the unique unambiguous truth vector $\mathbf{x}$.

A. Optimum number of measurements

In sparse reconstruction theory, a k-sparse unknown vector $\mathbf{x} \in \mathcal{R}^n$ can be reconstructed with the number of measurements given by $m_s=O(k\cdot\log_2(n/k))$ [8] using randomized matrix constructions with small restricted isometry constants. The Restricted Isometry Property (RIP) introduced by Candés and Tao in [9] is a widely used tool for analyzing the orthonormal properties of a measurement matrix required for sparse reconstruction via L^1 minimization. The RIP measures if the sensing matrix Φ preserves the Euclidean norm of sparse inputs to within two constants $(1\text{-}\delta)$ and $(1+\delta)$ when $\mathbf{x}$ is $2k$-sparse in order to ensure recovery of k-sparse $\mathbf{x}$. This is denoted by RIP $(2k, \delta)$ and mathematically stated as:

$$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|\Phi\mathbf{x}\|_2^2 \leq (1+\delta)\|\mathbf{x}\|_2^2. \quad (8)$$

Unfortunately, there is no deterministic approach to construct a measurement matrix with the required RIP, or to efficiently check if the RIP of a given measurement matrix has good recovery guarantees. It was shown in [10] that binary matrices cannot achieve good performance with respect to the RIP and hence suffer inherent limitations. However, it has been shown that such matrices, nevertheless, satisfy a different form of the RIP called RIP-p property [11], where the L^2 norm is replaced by the L^p norm. Also, it has been pointed out in [7] that "RIP conditions are only sufficient and often fail to characterize all good measurement matrices". The construction of explicit matrices with optimum number of measurements for sparse reconstruction is an active area of research in compressed sensing [12].

Berinde and Indyk [13] consider matrices that are binary and sparse, i.e. they have only a fixed small number of ones d in each column, and all the other entries are equal to zero. Experimental results showed that the binary sparse matrices are as "good" as the random Gaussian or Fourier matrices for L^1 recovery both in terms of necessary measurements and in terms of the recovery error. Another advantage of such matrices is their efficient update time, proportional to the sparsity parameter d.

B. Simulation Setup

A Matlab code was developed to simulate the ambiguity resolution and recovery of any k-sparse truth vector $\mathbf{x} \in \mathcal{B}^n$ using L^1 minimization, given the aliased radar measurements made using different PRFs. Several L^1 minimization libraries are publicly available to be used for the necessary computations. Reconstruction techniques differ in terms of the underlying model and the methods employed to solve the problem. For the case without any measurement errors, the linear model **Ax** = **b** can be used and solved for the minimum L^1 norm **x**, which is known as the *basis pursuit* method. The *basis pursuit denoising* method takes into account measurement errors in the system and solves problems of the form

$$\mathbf{x} = \min \|\mathbf{x}\|_1 \text{ subject to } \|\mathbf{Ax} - \mathbf{y}\|_2 \le \boldsymbol{\varepsilon}, \qquad (9)$$

where **ε** is a user specified estimate of the standard deviation of errors in the system. Another formulation for basis pursuit denoising is the *L^1 regularized least squares* problem defined by

$$\mathbf{x} = \min \|\mathbf{Ax} - \mathbf{y}\|_2 + \lambda \|\mathbf{x}\|_1, \qquad (10)$$

where $\lambda \ge 0$ is a regularization parameter that also governs the sparsity of the solution. Sparse solutions are obtained for sufficiently large values of λ [14], [15]. A Matlab based solver *l1_ls* [16] uses the interior point method to solve the L^1 regularized least squares problem. This code handles large sparse problems efficiently and has a variant that solves for strictly non-negative **x**, making it particularly well-suited to our application.

To investigate the effect of changing variables such as the sparsity *k*, number of PRFs, and the number of measurements m_s on the recovery of truth vector **x**, we followed the approach where *k* is taken to be a small fraction of *n* and the number of measurements is also a fraction of *n*. In a typical radar detection application, it is reasonable to assume target sparsity of the order of 1% of *n*.

C. Simulation Results

The L^1 minimization approach for ambiguity resolution was tested using radar parameters based on an X-band airborne pulse-Doppler medium PRF mode radar (see [17], Table I]. 8 targets were randomly distributed in the truth vector **x**, which consists of a total of 200,000 range-Doppler bins. Some random target locations may fall in range or Doppler blind zones for a particular PRI set, but for this proof of concept demonstration, it has been assumed that they are nonetheless detected. Extensions have been developed to account for missed detections due to blind zones.

We used the medium PRF set of 8 PRFs found using evolutionary algorithms for optimum blind zone performance with PRIs of {51, 53, 60, 63, 67, 84, 89, 93} µs [17]. From the PRF set and other radar parameters, we can calculate the number of range cells in the unambiguous range for each PRF, as well as the number of velocity cells in the unambiguous velocity distribution (0, PRF) of each PRF. Since a sampling time of 1µs is assumed in the radar parameters, the number of unambiguous range cells is **R** = {51, 53, 60, 63, 67, 84, 89, 93} and the number of unambiguous velocity cells is **V** = {196, 189, 167, 159, 149, 119, 112, 108} respectively for our 8 PRF set. Fig. 1 shows the actual target distribution in the 2-D range-Doppler matrix. For demonstration purposes, targets have been magnified from their actual girth of one bin to the magnified diamond shapes in Figs. 1-3. After the range-Doppler matrix is vectorized as explained in Section III, m_s is now given by the sum of the number of cells in the unambiguous range times the number of velocity cells in the unambiguous velocity measurements for each PRF, i.e. $m_s = \sum_{i=1}^{8} R_i V_i = 80041$. *n* becomes the number of unaliased range bins times the number of unaliased velocity cells (1000×200). Consequently, the m_s/n ratio in this case is 80041/200000 = 0.4 which means the total number of measurements is 40% of the total number of unknowns.

To demonstrate the ambiguity in the measured data, the aliased measurements made with three of the set of eight different PRIs, {51, 63, 93} µs, are shown in Fig. 2 (a), (b) and (c). The dotted rectangle indicates the extent of the range-Doppler cells covered with a single PRF measurement, i.e. all the measurements will be folded over in the area bounded by this rectangle. The problem is solved using the *l1_ls* Matlab solver and the solution is obtained within less than 10 seconds on a laptop computer. Fig. 3 shows the minimum L^1 solution of the system converted back to matrix form. It can be seen from Fig. 1 and Fig. 3 that the L^1 minimization accurately resolves all target ambiguities in both range and Doppler in the absence of measurement errors.

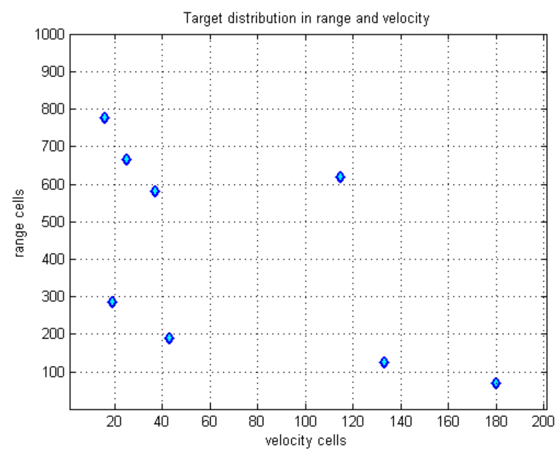

Fig. 1. Actual target distribution (range and Doppler).

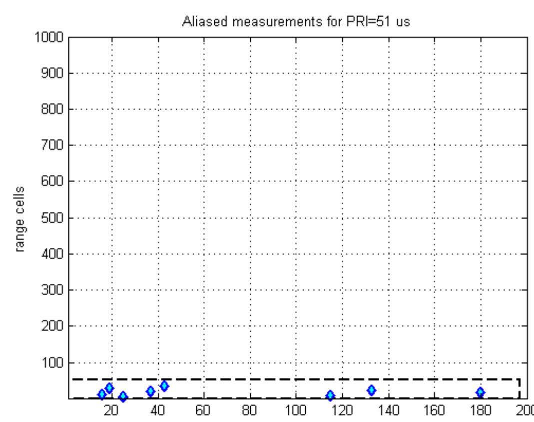

Fig. 2(a). Aliased measurement, PRI = 51 µs.

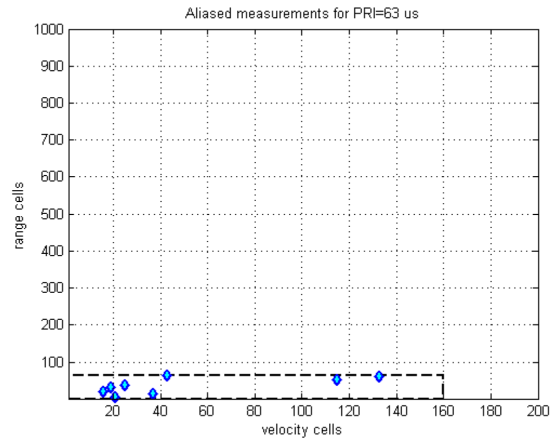

Fig. 2(b). Aliased measurement, PRI = 63 μs.

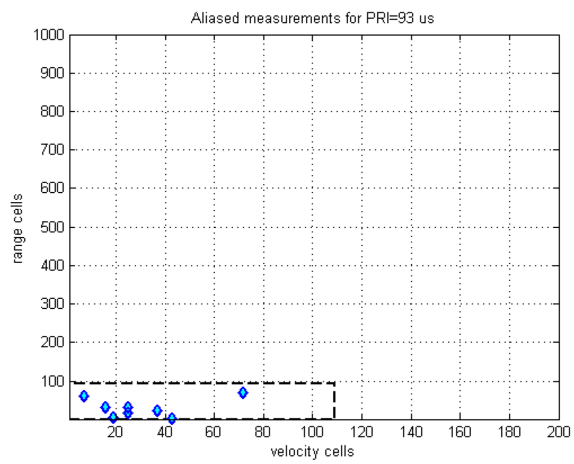

Fig. 2(c). Aliased measurement, PRI = 93 μs.

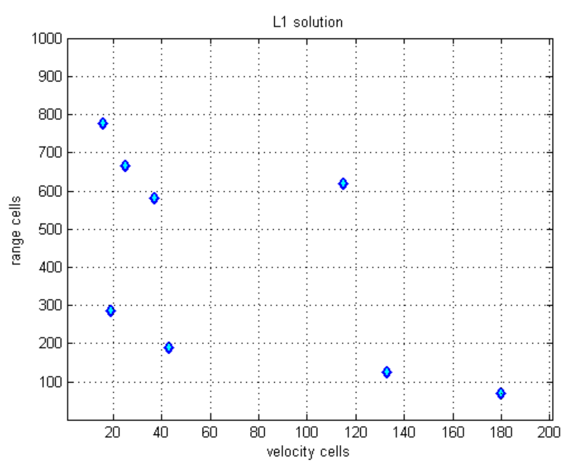

Fig. 3. L^1 recovery of the unambiguous target distribution.

It is important to note that the solution may be incorrect if the number of measurements is too small for a given selection of target sparsity k and the number of PRFs. In that case, the target energy starts spreading across multiple candidate range-Doppler cells in the solution vector that could account for the target location in the absence of a sufficient number of measurements for a definitive solution.

To demonstrate the dependence of the required number of measurements on k and the number of PRFs, Monte Carlo simulations were carried out for various number of measurements m_s in the range from 5% to 50% of n for a given set of n, k and the number of PRFs. With 100 trials for each value of m_s and the number of times the correct solution is obtained, a probability of correct solution can be defined. We define a "correct" solution as one having an L^2 norm less than or equal to 0.001 for the difference between the actual truth vector and the solution vector, which is a very conservative measure.

Fig. 4 shows that if k is a larger percentage of n (denser target environment), then the required m_s is also a larger percentage of n provided the number of PRFs remains constant. In terms of radar measurements, larger m_s with a fixed number of PRFs is equivalent to having longer PRIs. Alternatively, if m_s is kept fixed the probability of obtaining the correct solution increases with the number of PRFs which implies that more, shorter PRIs would constitute a better measurement structure compared to fewer, longer PRIs. This is shown in Fig. 5. Fig. 4 and Fig. 5 suggest having as many PRFs as allowed by the system requirements and constraints, and then setting the PRIs accordingly taking into account the minimum number of measurements required for the particular number of PRFs chosen for the system.

Current CRT-based ambiguity resolution techniques require detection in at least (N+1) PRFs to disambiguate N targets. Empirical results show that our technique can resolve multiple targets with fewer PRFs provided that the number of measurements is sufficient to obtain the correct solution. As shown in Fig. 4 and Fig. 5, ambiguity resolution of multiple targets depends on the total number of measurements as well as the number of PRFs. The greater the number of PRFs, the smaller the total number of measurements required to solve the problem.

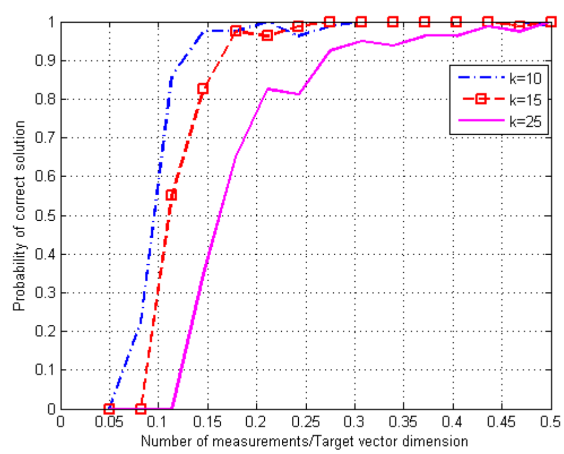

Fig. 4. Effect of varying target sparsity (n=1000, #PRFs =3).

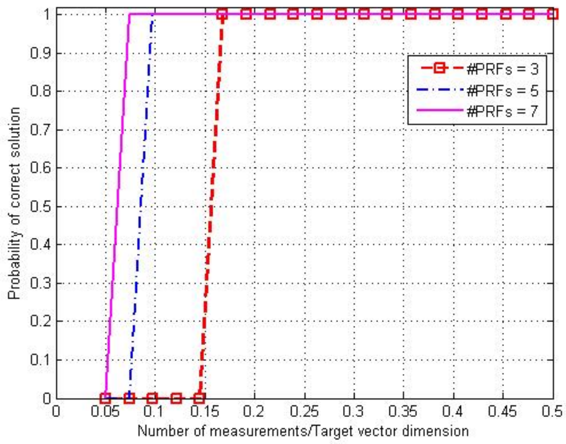

Fig. 5. Effect of varying the number of PRFs (n=1000, k=10).

V. CONCLUSIONS/FUTURE WORK

A new range/Doppler ambiguity resolution method is proposed based on the L^1 minimization of the system of equations representing the sparse detection data measured with different PRFs. The new technique can efficiently resolve ambiguities in range and Doppler and requires fewer PRFs than the CRT based techniques. Future work will continue ongoing investigations of the modeling of real-world data errors, their effect on the solution technique, and extensions to compensate for such errors. Examples of such errors include false alarms, missed detections, and blind zones.

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